# Math 255A' Lecture 22 Notes

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# 1 Extension of Functional Calculus and Proof of The Spectral Theorem

## 1.1 Proof of the spectral theorem for self-adjoint operators

So far, we've constructed continuous functional calculus: a map $C([a,b]) \to \mathcal{B}_{\mathrm{sa}}(H)$ sending $f \mapsto f(T)$ which is

- linear,
- $f(g)(T) = f(T)g(T)$,
- $f \geq g \implies f(T) \geq g(T)$,
- $1(T) = I$,
- $\|f(T)\| \leq \|f\|_{\sup}$.

If $(f_n)_n$ is a sequence in $C([a,b])$ with $f_n \geq 0$ and $f_n(x) \downarrow g(x)$ for all $x \in [a,b]$, then we want to define $g(T)$ by $\langle g(T)x, y\rangle = \lim_n \langle f_n(T)x, y\rangle$. Last time, we showed that this limit exists (as a weak operator topology limit).

**Lemma 1.1.** *Suppose $f_n, f_n' \downarrow g$. Then the limit, $g(T)$, is the same.*

*Proof.* Let $f_n, f_n' \downarrow g$. For every $x$, $\varepsilon > 0$, and $n \in \mathbb{N}$, there exists an $n'(x,\varepsilon)$ such that $f_{n'}'(x) < g(x) + \varepsilon \leq f_n(x) + \varepsilon$. Then for each $n \in \mathbb{N}$, $\varepsilon > 0$ and $x$, we get $n'(n,x,\varepsilon)$ and a neighborhood $U(n,x,\varepsilon)$ of $x$ such that $f_{u'}'|_{U_{n,x,\varepsilon}} < (f_n + \varepsilon)|_{U(n,x,\varepsilon)}$. Choose $x_1, \dots, x_t$ such that $\bigcup_{i=1}^t U(n, x_i, \varepsilon) = [a,b]$. Let $n'' = \max(n'(n,x_1,\varepsilon), \dots, n'(n,x_t,\varepsilon))$. Now $f_{n''}' < f_n + \varepsilon$ on $[a,b]$. Then $f_{n''}'(T) \leq f_n(T) + \varepsilon I$, so $\lim_{n''} f_{n''}'(T) \leq f_n(T) + \varepsilon$ for all $n, \varepsilon$. Since $\varepsilon$ is arbitrary, and by symmetry, we get that $\lim f_n'(T) \leq \lim_n f_n(T)$ and $\lim f_n'(T) \geq \lim_n f_n(T)$. So the limits are equal. $\square$

Now, if we have $f_n \downarrow g \geq 0$, we get $g(T) \geq 0$. This is

- still additive: If $f_n \downarrow g$, $f'_n \downarrow g$;, then $f_n + f'_n \downarrow g + g'$. We have

$$(g+g')(T) = \mathrm{WO}\lim_n (f_n(T) + f'_n(T)) = g(T) + g'(T).$$

**Lemma 1.2.** *If $f_n \downarrow g \geq 0$, and $f'_n \downarrow g'_n \geq 0$, then*

$$(gg')(T) = g(T)g'(T).$$

*Proof.* We have $f_n f'_n \downarrow gg'$, so $(gg')(T) = \mathrm{WO}\lim_n (f_n f'_n)(T)$. We want to show that the is the product of the limits of $f_n(T)$ and $f'_n(T)$. By polarization, it is enough to show that $\lim_n \langle (f_n f'_n)Tx, x\rangle = \lim_n \lim_m \langle f_n(T) f'_m(T)x, x\rangle$. The limit of the diagonal terms is the same as $\lim_n \lim_m$ because the array is decreasing in $n, m$ (a basic real analysis fact). $\square$

Given $\lambda \, in[a,b]$ and $n \in \mathbb{N}$, define

$$\varphi_n^\lambda(t) = \begin{cases} 1 & t \leq \lambda \\ -n(x - (\lambda + 1/n)) & \lambda < t \leq \lambda + 1/n \\ 0 & t > \lambda + 1/n \end{cases}$$

Then $\varphi_n^\lambda \downarrow \mathbb{1}_{(-\infty,\lambda]}$ as $n \to \infty$. Define $E(\lambda) := \lim_n \varphi_n^\lambda(T)$.

Here are the properties of $E(\lambda)$:

1. $E(\lambda)$ is self adjoint (as a WO limit of self-adjoints).
2. $E(\lambda) = \mathbb{1}_{(-\infty,\lambda]}(T) = \mathbb{1}^2_{(-\infty,\lambda]}(T) = E(\lambda)^2$.
3. If $\lambda \geq \mu$, then

$$E(\mu)E(\lambda) = E(\lambda)E(\mu) = (\mathbb{1}_{(-\infty,\lambda]}\mathbb{1}_{(-\infty,\mu]})(T) = E(\mu).$$

4. Declare $E(\lambda) = 0$ if $\lambda < a$ and $E(b) = \lim_n 1(T) = I$.
5. Fix $\lambda \in [a,b]$. Then $E(\mu)x \to E(\lambda)x$ as $\mu \downarrow \lambda$ for all $x \in H$. Equivalently, $\langle (E(\mu) - E(\lambda))x, x\rangle \to 0$.

   To show this, we know $\langle E(\lambda)x, x\rangle = \lim_n \left\langle \varphi_n^\lambda(T)x, x\right\rangle$. Pick $n$ large enough so that $\left\langle \varphi_n^\lambda(T)x, x\right\rangle < \langle E(\lambda)x, x\rangle + \varepsilon$. This is also $\lim_{\mu \downarrow \lambda} \langle \varphi_n^\mu(T)x, x\rangle$. So for $\mu$ close enough to $\lambda$, we get

$$\langle E(\mu)x, x\rangle \leq \langle \varphi_n^\mu(T)x, x\rangle < \langle \varphi_n^\lambda(T)x, x\rangle + \varepsilon < \langle E(\lambda)x, x\rangle + 2\varepsilon.$$

This gives us a spectral family for $T$. If $a \leq \mu \leq \lambda \leq b$, then

$$E(\mu, \lambda] := E(\lambda) - E(\mu) = \mathrm{WO}\lim_n [\varphi_n^\lambda(T) - \varphi_n^\mu(T)].$$

This gives us

$$TE(\mu,\lambda] = \mathrm{WO}\lim_n T[\varphi_n^\lambda(T) - \varphi_n^\mu(T)] = \mathrm{WO}\lim_n[(t\cdot(\varphi_n^\lambda(t) - \varphi_n^\mu(t)))(T)].$$

Now

$$\mu\mathbb{1}_{(\mu+1/n,\lambda]} \le t(\varphi_n^\lambda(t) - \varphi_n^\mu(\lambda)) \le \lambda\mathbb{1}_{(\mu,\lambda+1/n]}$$

Taking the weak operator limit, we get

$$\mu E(\mu,\lambda) \le TE(\mu,\lambda) \le \lambda E(\mu,\lambda).$$

Now let $a = \lambda_0 < \lambda_1 < \cdots < \lambda_m = b$. Then

$$\begin{aligned} I &= E(B) \\ &= (E(\lambda_n) - E(\lambda_{n-1})) + \cdots + (E(\lambda_1) - E(\lambda_0)) \\ &= E(a,\lambda_1] + E(\lambda_1,\lambda_2] + \cdots + E(\lambda_{n-1},b]. \end{aligned}$$

Multiplying by $T$, we get

$$T = TE(a,\lambda_1] + TE(\lambda_1,\lambda_2] + \cdots + TE(\lambda_{n-1},b].$$

So we get

$$\sum_{i=1}^m \lambda_{i-1}E(\lambda_{i-1},\lambda_i] \le T \le \sum_{i=1}^n \lambda_i E(\lambda_{i-1},\lambda_i].$$

This gives

$$\sum_{i=1}^m \lambda_{i-1}\langle E(\lambda_{i-1},\lambda_i]x,x\rangle \le \langle Tx,x\rangle \le \sum_{i=1}^n \lambda_i\langle E(\lambda_{i-1},\lambda_i]x,x\rangle.$$

These are partial sums in the definition of the Riemann-Stieltjes integral. So taking the limit as $\max_i|\lambda_i - \lambda_{i-1}| \to 0$, we get

$$\langle Tx,x\rangle = \int \lambda\, d\langle E(\lambda)x,x\rangle.$$

This completes the proof of the spectral theorem.

### 1.2 Borel functional calculus and spectral measure

How far can we take this functional calculus? Here is another method which allows us to extend to all Borel functions. Assume we have a continuous functional calculus: $f \mapsto f(T)$ for all $f \in C([a,b])$. Given $x,y \in H$, consider

$$f \mapsto \langle f(T)x,y\rangle.$$

This is bounded by $|\langle f(T), x, y\rangle| \le \|f\|_{\sup}\|x\|\|y\|$. So there exists some $\mu_{x,y} \in M([a,b])$ such that $\|\mu_{x,y}\| \le \|x\|\|y\|$ and $\langle f(T)x, y\rangle = \int f\, d\mu_{x,y}$. So given $g$ bounded and Borel, define

$$Q_g(x,y) := \int g\, d\mu_{x,y}.$$

This is bilinear in $x, y$ and bounded: $|Q_g(x,y)| \le \|g\|_\infty \|x\|\|y\|$. At each step, our construction is symmetric in $x, y$, so $Q_g(x,y)$ is symmetric in $x, y$. Now define $g(T)$ by $\langle g(T)x, y\rangle = Q_g(x,y)$. We can now define, as before, $\mathbb{1}_{(-\infty,\lambda]}(T)$.

The advantage of this method is that we can also define $E(A) := \mathbb{1}_A(T)$ for all $A \in \mathcal{B}([a,b])$. We can now show that

- Every $E(A)$ is a projection.
- $E(A \cap B) = E(A)E(B)$.
- $E(\varnothing) = 0$, and $E([a,b]) = I$.
- $E(\bigcup_n A_n) = \sum_n E(A_n)$.

This gives a **spectral measure**, which has the properties of a measure but takes values in projections. More advanced versions of the spectral theorem use this approach.